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Exactly Complete Solutions of the Pseudoharmonic Potential in N-Dimensions

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Abstract We present analytically the exact solutions of the Schrödinger equation in the N-dimensional spaces for the pseudoharmonic oscillator potential by means of the ansatz method. The energy eigenvalues of the bound states are easily calculated from this eigenfunction ansatz. The normalized wavefunctions are also obtained. A realization of the ladder operators for the wavefunctions is studied and we deduced that these operators satisfy the commutation relations of the generators of the dynamical group $SU(1,1)$. Some expectation values for $\langle r^{-2}\rangle$, $\langle r^{2}\rangle$, $\langle T\rangle$, $\langle V\rangle$, $\langle H\rangle$, $\langle p^{2}\rangle$ and the virial theorem for the pseudoharmonic oscillator potential in an arbitrary number of dimensions are obtained by means of the Hellmann–Feynman theorems. Each solution obtained is dimensions and parameters dependent.

Keywords N-dimensions · Pseudoharmonic oscillator potential · Schrödinger equation · Exact solutions · Hyperspherical harmonics · Wavefunction ansatz · Ladder operators · SU(1, 1) · Expectation values · Hellmann–Feynman theorems · Virial theorems

1 Introduction

Solvable potentials play a very significant role in various fields of Physics. These potentials served as useful aids or tools in modeling realistic physical problems, and offered an interesting field of investigation in different fields of applications in Physics. The solution of the

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Schrödinger equation for any spherically symmetric potential in N-dimensional space has attracted much attention since the early study of non-relativistic quantum mechanics.

The extension of physical problems to higher-dimensional space or N-dimensional space plays an important role in many areas of Physics. The mathematical tools for generalization of the orbital angular momentum in an arbitrary dimensional space have been discussed [21, 41, 70]. The solutions of some physical systems can be obtained by solving the transformed Schrödinger equation in N-dimensions with the potential under consideration.

Some of the physical systems of interest in quantum mechanics that have been thoroughly studied in N-dimensional space are the two common exactly solvable potentials, these are, the harmonic and Coulomb potentials [4, 16, 19, 21, 22, 25, 26, 28, 64, 67, 77, 78, 84, 87, 91, 108, 111].

Recently, there has been much attention on the investigation of the dependence of physical systems upon the N-dimensional space. Among these are the investigation carried out on the relationship between a Coulomb and a harmonic oscillator potentials in arbitrary dimensions by [16, 19, 22, 25, 28, 64, 67, 77, 78, 91, 108, 111], followed by comments of ([91] and [77]) on the work of [16].

Al-Jaber [2–6] studied some aspects of the quantization of angular momentum; the radial-part equation of the wavefunction; the hydrogen atom; the Fermi gas and the Planck's spectral distribution laws respectively, in N-dimensional space. A unified treatment of screening Coulomb and anharmonic oscillator potentials, and the supersymmetric and the relationship between a class of singular potentials in arbitrary dimensions have been studied by Gönül et al. [48, 49]. Paz [87] discussed the connection between the radial momentum operator and the Hamiltonian in N-dimensions.

Fukutaka and Kashiwa [46], Neves and Wotzasek [82], Grosche and Steiner [52–54] considered the formulation of path integral and its quantization in N-dimensional space. So also, the N-dimensional Coulomb problems: Stark effect in hyperparabolic and hyperspherical coordinates and the relation between dimension and angular momentum for radially symmetric potential in N-dimensional space have been considered by [97, 104]. Chen [26] obtained the exact solutions of N-dimensional harmonic oscillator via Laplace transformation. Recent works in mathematical physics have been reported: The algebraic method in which group theory approach has been used in an arbitrary D-dimensional space by [18, 27, 55, 56, 65, 67], supersymmetry approach in an arbitrary D-dimensional space by [65, 66, 68, 106], quantum gravity theories in extra dimensions by [8, 17, 81].

Furthermore, the study of the Schrödinger equation with certain central physical potentials in D dimensions has become an important aspect in Quantum Mechanics, however, such a problem has been extended to the relativistic equation case by [31, 34, 55, 56]. Dong [30] applied a suitable ansatz to obtain the solutions of the D-dimensional radial Schrödinger equation with some anharmonic potentials. Dong and Ma [36] studied the non-relativistic Levinson's theorem in D dimensions, Dong et al. [38] used the group theory approach to study the Dirac equation with a Coulomb plus scalar potential in $D+1$ dimensions.

Oyewumi and Bangudu [83] discussed some aspects of the N-dimensional isotropic harmonic oscillator plus inverse quadratic potential. Also, Oyewumi and Ogunsola [84] considered the exact solutions of the harmonic oscillator in multidimensional spaces. Recently, Oyewumi [85] considered the analytic solutions of the Kratzer–Fues potential in an arbitrary number of dimensions. The entropy information, the Heisenberg and Fisher-information-based uncertainty relations for the D-dimensional central problems have been studied by [29, 92, 96, 109].

Chen [26] obtained the exact solutions of N-dimensional harmonic oscillator via Laplace transformation, also, the eigenfunctions and eigenvalues of the D-dimensional pseudoharmonic oscillator has been obtained by Wang et al. [103]. Al-Jaber and Lombard [7] investigated the connection between the spectrum and moments of the ground-state density in N-dimensional space. The exact polynomial solutions of the Mie-type and some molecular potentials in the N-dimensional Schrödinger equation have been obtained by Ikhdair and Sever [58–62]. Exact solutions of the Schrödinger equation in N-dimensions for: the pseudoharmonic potential plus ring-shaped potential; the modified Kratzer potential plus ring-shaped and the pseudo-Coulomb potential plus ring-shaped potential have been obtained by means of the Nikiforov–Uvarov method by Ikhdair and Sever [59–62]. Finally, generalized hypervirial theorem for the D-dimensional single-particle system and its applications have been studied by Ma and Dong [76].

Pseudoharmonic oscillator potential was first pointed out by Gol'dman et al. 1960 [47]. This potential has been studied in some detail by many authors due to its importance in Chemical Physics, Molecular Physics and other fields of Physics by ([31, 37, 39, 76] and the references therein; [42–44, 55, 56, 58–60, 62, 98–100, 105]. Sage and Goodisman [98] considered the improvement on the conventional presentation of molecular vibrations where the advantages of the pseudoharmonic potential and direct construction of potential energy curve were discussed. The creation and annihilation operators and the corresponding coherent state of the pseudoharmonic oscillator potential were obtained by [88]. Dong [30] discussed the realization of dynamic group structure for the pseudoharmonic oscillator potential. In the same way, B-G, K-P and G-K coherent states for this potential and some of their properties were examined by [90].

It is the purpose of this paper to obtain the quantum mechanical properties of the exactly complete solutions of the pseudoharmonic oscillator potential in N-dimensions. This work is organized as follows. Section 2 contains the exact-bound state solutions of the pseudoharmonic oscillator potential in arbitrary dimensions. Construction of the Ladder operators for the pseudoharmonic oscillator potential in N-dimensions is contained in Sect. 3. The expectation values for $\langle r^{-2}\rangle$, $\langle r^2\rangle$, $\langle T\rangle$, $\langle V\rangle$, $\langle H\rangle$, $\langle p^2\rangle$ and the virial theorem for this potential in N dimensions obtained by means of the Hellmann–Feynman theorems are contained in Sect. 4. Section 5 contains conclusion.

2 The Exact-Bound State Solutions of the Pseudoharmonic Oscillator Potential in N-Dimensions

2.1 Pseudoharmonic Oscillator Potential in N-Dimensions

Consider the pseudoharmonic oscillator potential ([30, 31, 39] and the references therein; [42–44, 47, 55, 56, 58–62, 88, 90, 98–100, 105])

$$V(r)=\frac{1}{8}kr_0^2\left(\frac{r}{r_0}-\frac{r_0}{r}\right)^2, \tag{1}$$

where r_0 is the equilibrium bond length and k the force constant.

The eigenvalue equation for the pseudoharmonic oscillator potential in an arbitrary number of dimensions is

$$\left[\frac{-\hbar^2}{2\mu}\Delta_N+\frac{1}{8}kr_0^2\left(\frac{r}{r_0}-\frac{r_0}{r}\right)^2\right]\Psi(r,\Omega_N)=E\Psi(r,\Omega_N), \tag{2}$$

since the hyperspherical harmonics are eigenfunctions of the operator $\Lambda_N^2(\Omega_N)$, we can therefore seek for the solutions of the Schrödinger equation in the form

$$\Psi_{n_r,\ell,m}(r,\Omega_N)=R_{n_r,\ell}(r)Y_\ell^m(\Omega_N). \tag{3}$$

With equation (3), Equation (2) reduces to two separate equations:

$$R''_{n_r,\ell}(r)+\frac{N-1}{r}R'_{n_r,\ell}(r)-\ell(\ell+N-2)\frac{R_{n_r,\ell}(r)}{r^2} \\ +\frac{2\mu}{\hbar^2}\left[E-\frac{1}{8}kr_0^2\left(\frac{r}{r_0}-\frac{r_0}{r}\right)^2\right]R_{n_r,\ell}(r)=0, \tag{4}$$

$$\Lambda_N^2(\Omega_N)Y_\ell^m(\Omega_N)-\ell(\ell+N-2)Y_\ell^m(\Omega_N)=0. \tag{5}$$

The solutions of the hyperangular part equation (5) is the hyperspherical harmonics $Y_\ell^m(\Omega_N)$ [4, 9, 10, 53, 54, 70, 93]. $R_{n_r,\ell}(r)$ is the hyperradial part of the problem, E is the energy eigenvalue, and ℓ is "orbital angular momentum" quantum number. Equation (4) is called the hyperradial Schrödinger equation for the pseudoharmonic potential.

Using the dimensionless abbreviations:

$$K^2=\frac{2\mu E}{\hbar^2} \tag{6}$$

(4) becomes

$$R''_{n_r,\ell}(r)+\frac{N-1}{r}R'_{n_r,\ell}(r)+\left[K^2+\frac{\mu kr_0^2}{2\hbar^2}-\frac{\mu kr^2}{4\hbar^2}-\frac{\mu_\ell(\mu_\ell+N-2)}{r^2}\right]R_{n_r,\ell}(r)=0, \tag{7}$$

where

$$\mu_\ell=\frac{1}{2}\left[\frac{-(N-2)}{2}+\sqrt{(2\ell+N-2)^2+\frac{\mu k}{\hbar^2}r_0^4}\right]. \tag{8}$$

According to the asymptotic behaviors of the radial wave functions as $r\to 0$ and $r\to\infty$, the physically acceptable solution of $R_{n_r,\ell}(r)$ can be expressed as

$$R_{n_r,\ell}(r)=Cr^{\mu_\ell}e^{-\lambda r^2}f(r). \tag{9}$$

Substituting this into (7) yields

$$f''(r)+\left[\frac{(2\mu_\ell+N-1)}{r}-4\lambda r\right]f'(r) \\ +\left[\frac{\mu kr_0^2}{2\hbar^2}-\frac{\mu kr^2}{4\hbar^2}-2\lambda(2\mu_\ell+N)+4\lambda^2r^2+K^2\right]f(r)=0, \tag{10}$$

now putting

$$4\lambda^2=\frac{\mu k}{4k^2}\quad\Rightarrow\quad\lambda=\frac{1}{4\hbar^2}\sqrt{\mu k} \tag{11}$$

(11) becomes

$$f''(r)+\left[\frac{(2\mu_\ell+N-1)}{r}-\sqrt{\frac{\mu k}{\hbar^2}}r\right]f'(r)$$

$$-\left[\frac{1}{2}(2\mu_\ell+N)\sqrt{\frac{\mu k}{\hbar^2}}-\frac{\mu k r_0^2}{2\hbar^2}-K^2\right]f(r)=0, \tag{12}$$

with

$$\rho=2\lambda r^2 \tag{13}$$

(11) becomes

$$\rho f''(\rho)+\left[\frac{(2\mu_\ell+N)}{2}-\rho\right]f'(\rho)$$
$$-\left[\frac{(2\mu_\ell+N)}{4}-\sqrt{\frac{\mu k}{16\hbar^2}}r_0^2-\frac{1}{2}\sqrt{\frac{\hbar^2}{\mu k}}K^2\right]f(\rho)=0. \tag{14}$$

Equation (14) is the Kummer's (confluent hypergeometric) differential equation and the solution of this equation that is regular at $r=0$ or $\rho=0$ is the degenerate hypergeometric function or the Kummer's function [1]

$$f(\rho)={}_1F_1\left[\frac{1}{4}\left((2\mu_\ell+N)-r_0^2\sqrt{\frac{\mu k}{\hbar^2}}-2K^2\sqrt{\frac{\hbar^2}{\mu k}}\right),\frac{2\mu_\ell+N}{2};\rho\right], \tag{15}$$

for large values of ρ, the solution in (15) diverges as $\exp(\sqrt{\frac{\mu k}{16\hbar^2}}r^2)$, thus preventing normalization, except for

$$n_r=\frac{1}{4}\left[2K^2\sqrt{\frac{\hbar^2}{\mu k}}+r_0^2\sqrt{\frac{\mu k}{\hbar^2}}-2\mu_\ell-N\right];\quad n_r=0,1,2,\dots, \tag{16}$$

which gives the energy eigenvalues as:

$$E_{n_r}=\left(n_r+\frac{2\mu_\ell+N}{4}\right)\sqrt{\frac{\hbar^2 k}{\mu}}-\frac{kr_0^2}{4}=\frac{1}{2}\left(n+\frac{N}{2}\right)\sqrt{\frac{\hbar^2 k}{\mu}}-\frac{kr_0^2}{4}. \tag{17}$$

Therefore the unnormalized eigenfunction of the hyperradial Schrödinger equation with the pseudoharmonic oscillator potential is:

$$R_{n_r,\ell}(r)=C_{n_r,\ell}r^{\mu_\ell}\exp\left(-\sqrt{\frac{\mu k}{16\hbar^2}}r^2\right){}_1F_1\left(-n_r,\frac{2\mu_\ell+N}{2};\sqrt{\frac{\mu k}{4\hbar^2}}\right). \tag{18}$$

The normalization constant

$$C_{n_r,\ell}=\left[\frac{2(\frac{\mu k}{4\hbar^2})^{\frac{1}{4}(2\mu_\ell+N)}n_r!}{\Gamma(n_r+\mu_\ell+\frac{N}{2})}\right]^{\frac{1}{2}}\frac{(\mu_\ell+n_r+\frac{N-2}{2})!}{n_r!(\mu_\ell+\frac{N-2}{2})} \tag{19}$$

is determined from the requirement that:

$$\int_0^\infty |R_{n_r,\ell}(t)|^2 r^{N-1}dr=1, \tag{20}$$

and

$$ {}_1F_1(-\gamma, m+1; Z) = \frac{\gamma! m!}{(\gamma+m)!} L_\gamma^m(Z), \tag{21} $$

where $L_\gamma^m(Z)$ are associated Laguerre polynomials.

Therefore, (18) becomes

$$ R_{n_r,\ell}(r) = \left[\frac{2(\frac{\mu k}{4\hbar^2})^{\frac{1}{4}(2\mu_\ell+N)} n_r!}{\Gamma(n_r+\mu_\ell+\frac{N}{2})}\right]^{\frac{1}{2}} r^{\mu_\ell} \exp\left(-\sqrt{\frac{\mu k}{16\hbar^2}} r^2\right) L_{n_r}^{\mu_\ell+\frac{N-2}{2}}\left(\sqrt{\frac{\mu k}{4\hbar^2}} r^2\right), \tag{22} $$

and the corresponding eigenvalue is

$$ E_{n_r} = \left(n_r + \frac{2\mu_\ell+N}{4}\right)\sqrt{\frac{\hbar^2 k}{\mu}} - \frac{k r_0^2}{4} = \frac{1}{2}\left(n+\frac{N}{2}\right)\sqrt{\frac{\hbar^2 k}{\mu}} - \frac{k r_0^2}{4}. \tag{23} $$

Therefore, the complete energy eigenfunctions of this potential in N-dimensional space is,

$$ \Psi_{n_r,\ell,m}(r, \Omega_N) = C_{n_r,\ell} R_{n_r,\ell}(r) Y_\ell^m(\Omega_N). \tag{24} $$

Where $C_{n_r,\ell}$ and $R_{n_r,\ell}(r)$ are as given by (19) and (22), respectively and $Y_\ell^m(\Omega_N)$ are the hyperspherical harmonics as earlier mentioned.

3 Construction of the Ladder Operators

We shall now construct the creation and annihilation operators corresponding to the pseudo-harmonic eigenfunction of (22) with the factorization method. In the recent years, factorization method [15, 23, 24, 31–36, 39, 63, 69, 72–75, 94, 101, 106, 107] has received a renewed attention because of the applications of these algebraic methods to quantum mechanical problems.

Among these applications that find their relevance to our work in this section are: the work of Zeng et al. [110], where an algebraic $SU(1,1)$ dynamical approach has been used to find the most general and simplest algebraic relationship between energy eigenstates of a hydrogen atom and a harmonic oscillator of arbitrary dimensions; Wipf et al. [106] obtained the algebraic solution of the supersymmetric Hydrogen atom; Martínez and Romero et al. [72–75] obtained the algebraic $SU(1,1)$ solutions for the hydrogen and relativistic hydrogen atom. The ladder operators can be generated directly from the eigenfunction and we shall find differential operators $\hat{L}_\pm$ with the following property

$$ \hat{L}_\pm R_{n_r,\ell}(r) = \ell_\pm R_{n_r\pm1,\ell}(r). \tag{25} $$

The operators of the form

$$ \hat{L}_\pm = A_\pm(r)\frac{d}{dr} + B_\pm(r) \tag{26} $$

which depend only on the physical variable r are to be obtained.

We can rewrite the normalized wavefunction in (22) as

$$ R_{n_r,\ell}(r) = N_{n_r}^\lambda r^{\mu_\ell} \exp(-\lambda r^2) L_{n_r}^{\mu_\ell+\frac{N-2}{2}}(2\lambda r^2), \tag{27} $$

where

$$N_{n_r}^{\lambda}=\left[\frac{2(2\lambda)^{(2\mu_\ell+N)/2}n_r!}{(n_r+\mu_\ell+\frac{N}{2})!}\right]^{\frac{1}{2}}. \tag{28}$$

The annihilation operator can be obtained as (see Appendix 2)

$$\hat{\mathcal{L}}_-=\frac{1}{2}\left[-r\frac{d}{dr}-2\lambda r^2+2n_r+\mu_\ell\right], \tag{29}$$

operation of this annihilation operator on the radial wavefunctions $R_{n_r,\ell}(r)$ allows us to obtain:

$$\hat{\mathcal{L}}_-R_{n_r,\ell}(r)=\ell_-R_{n_r-1,\ell}(r), \tag{30}$$

where

$$\ell_-=\sqrt{n_r\left(n_r+\mu_\ell+\frac{N-2}{2}\right)}. \tag{31}$$

The corresponding creation operator can also be obtained in a similar fashion as (see Appendix 2):

$$\hat{\mathcal{L}}_+=\frac{1}{2}\left[r\frac{d}{dr}-2\lambda r^2+2n_r+\mu_\ell+N\right], \tag{32}$$

operation of this creation operator on the radial wavefunctions $R_{n_r,\ell}(r)$ gives:

$$\hat{\mathcal{L}}_+R_{n_r,\ell}(r)=\ell_+R_{n_r+1,\ell}(r), \tag{33}$$

with

$$\ell_+=\sqrt{(n_r+1)\left(n_r+\mu_\ell+\frac{N}{2}\right)}. \tag{34}$$

In studying the dynamical group associated to the annihilation and creation operators $\hat{\mathcal{L}}_-$ and $\hat{\mathcal{L}}_+$, based on the results of (30) and (33) the commutator $[\hat{\mathcal{L}}_-,\hat{\mathcal{L}}_+]$ can be evaluated as follows:
Since $\hat{\mathcal{L}}_\mp$ operate on $R_{n_r,\ell}(r)$ respectively to give

$$\hat{\mathcal{L}}_-R_{n_r,\ell}(r)=\sqrt{n_r\left(n_r+\mu_\ell+\frac{N-1}{2}\right)}R_{n_r-1,\ell}(r) \tag{35}$$

and

$$\hat{\mathcal{L}}_+R_{n_r,\ell}(r)=\sqrt{(n_r+1)\left(n_r+\mu_\ell+\frac{N}{2}\right)}R_{n_r+1,\ell}(r), \tag{36}$$

then, the commutator of $\hat{\mathcal{L}}_-$ and $\hat{\mathcal{L}}_+$ gives (see Appendix 2)

$$[\hat{\mathcal{L}}_-,\hat{\mathcal{L}}_+]R_{n_r,\ell}(r)=2\ell_0R_{n_r,\ell}(r), \tag{37}$$

where we have introduced the eigenvalue

$$\ell_0=n_r+\frac{2\mu_\ell+N}{4}. \tag{38}$$

Operator $\hat{\mathcal{L}}_0$ can be defined as

$$\hat{\mathcal{L}}_0 = \hat{n}_r + \frac{2\mu_\ell + N}{4}. \tag{39}$$

Operators $\hat{\mathcal{L}}_\mp$ and $\hat{\mathcal{L}}_0$ satisfy the following commutation relations (see Appendix 2):

$$[\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_-]R_{n_r,\ell}(r) = -\hat{\mathcal{L}}_- R_{n_r,\ell}(r), \tag{40}$$

and also

$$[\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_+]R_{n_r,\ell}(r) = \hat{\mathcal{L}}_+ R_{n_r,\ell}(r). \tag{41}$$

The operators $\hat{\mathcal{L}}_\mp$ and $\hat{\mathcal{L}}_0$ satisfy the commutation relations of the dynamical group $SU(1,1)$ algebra, which is isomorphic to an $SO(2,1)$ algebra (i.e. $SU(1,1) \sim SO(2,1)$ ([15] and reference therein).

The Hamiltonian operator $\hat{\mathcal{H}}$ takes the form

$$\hat{\mathcal{H}} R_{n_r,\ell}(r) = \hat{\mathcal{L}}_0 \left(\frac{\hbar^2 k}{\mu}\right)^{1/2} - \frac{k r_0^2}{4} R_{n_r,\ell}(r). \tag{42}$$

The following expressions can easily be obtained from the operators $\hat{\mathcal{L}}_\mp$ and $\hat{\mathcal{L}}_0$ as follows (see Appendix 2):

$$\{\hat{\mathcal{L}}_0(\hat{\mathcal{L}}_0 - 1) - \hat{\mathcal{L}}_+\hat{\mathcal{L}}_-\}R_{n_r,\ell}(r) = \gamma(\gamma - 1)R_{n_r,\ell}(r) \tag{43}$$

and

$$\{\hat{\mathcal{L}}_0(\hat{\mathcal{L}}_0 + 1) - \hat{\mathcal{L}}_-\hat{\mathcal{L}}_+\}R_{n_r,\ell}(r) = \gamma(\gamma - 1)R_{n_r,\ell}(r). \tag{44}$$

Therefore the Casimir operator [20] can be expressed as

$$\begin{aligned} \hat{C} R_{n_r,\ell}(r) &= \{\hat{\mathcal{L}}_0(\hat{\mathcal{L}}_0 - 1) - \hat{\mathcal{L}}_+\hat{\mathcal{L}}_-\}R_{n_r,\ell}(r) = \{\hat{\mathcal{L}}_0(\hat{\mathcal{L}}_0 + 1) - \hat{\mathcal{L}}_-\hat{\mathcal{L}}_+\}R_{n_r,\ell}(r) \\ &= \gamma(\gamma - 1)R_{n_r,\ell}(r). \end{aligned} \tag{45}$$

Similarly, the following expressions can be easily obtained from the operators $\hat{\mathcal{L}}_\mp$ and $\hat{\mathcal{L}}_0$ as follows:

$$r^2 R_{n_r,\ell}(r) = \frac{1}{2\lambda}[2\hat{\mathcal{L}}_0 - (\hat{\mathcal{L}}_+ + \hat{\mathcal{L}}_-)]R_{n_r,\ell}(r), \tag{46}$$

$$r\frac{d}{dr} R_{n_r,\ell}(r) = (\hat{\mathcal{L}}_+ - \hat{\mathcal{L}}_-) - \frac{N}{2} R_{n_r,\ell}(r), \tag{47}$$

from which, on using (35), (36), (38) and (39), (see Appendix 2) we have

$$\begin{aligned} &\langle R_{m_r,\ell}(r)|r^2|R_{n_r,\ell}(r)\rangle \\ &\quad = \frac{1}{2\lambda}\left[\left(2n_r + \frac{2\mu_\ell + N}{2}\right)\delta_{m_r,n_r} - \ell_+\delta_{m_r,n_r+1} - \ell_-\delta_{m_r,n_r-1}\right] \end{aligned} \tag{48}$$

and

$$\langle R_{m_r,\ell}(r)|r\frac{d}{dr}|R_{n_r,\ell}(r)\rangle = \ell_+\delta_{m_r,n_r+1} - \ell_-\delta_{m_r,n_r-1} - \frac{N}{2}\delta_{m_r,n_r}. \tag{49}$$

We can deduce the following relations from (48) and (49):

$$
\begin{aligned}
&2\lambda\langle R_{m_r,\ell}(r)|r^2|R_{n_r,\ell}(r)\rangle + \langle R_{m_r,\ell}(r)|r\frac{d}{dr}|R_{n_r,\ell}(r)\rangle \\
&\quad = (2n_r + \mu_\ell)\delta_{m_r,n_r} - 2\ell_-\delta_{m_r,n_r-1}
\end{aligned}
\tag{50}
$$

and

$$
\begin{aligned}
&2\lambda\langle R_{m_r,\ell}(r)|r^2|R_{n_r,\ell}(r)\rangle - \langle R_{m_r,\ell}(r)|r\frac{d}{dr}|R_{n_r,\ell}(r)\rangle \\
&\quad = (2n_r + \mu_\ell + N)\delta_{m_r,n_r} - 2\ell_+\delta_{m_r,n_r+1}.
\end{aligned}
\tag{51}
$$

These relations form a useful link for finding the matrix elements from ladder operators.

4 Calculation of Some Useful Expectation Values and the Virial Theorem

Some useful expectation values $\langle r^2\rangle$, $\langle r^{-2}\rangle$, $\langle\hat{\mathcal{H}}\rangle$, $\langle V\rangle$, $\langle T\rangle$, $\langle P^2\rangle$ and the virial theorem for the pseudoharmonic potential in an arbitrary number of dimensions can be obtained by applying Hellmann–Feynman theorem (HFT) [4, 7, 11–13, 20, 40, 45, 51, 57, 71, 80, 85, 86, 89, 95].

Assuming that the Hamiltonian $\hat{\mathcal{H}}$ for a particular quantum mechanical system is a function of some parameter q, let $E(q)$ and $\Psi(q)$ be the eigenvalues and the eigenfunctions of the Hamiltonian $\hat{\mathcal{H}}(q)$. Then, the Hellmann–Feynman theorem states that

$$
\frac{\partial E(q)}{\partial q} = \langle\Psi(q)|\frac{\partial\hat{\mathcal{H}}(q)}{\partial q}|\Psi(q)\rangle.
\tag{52}
$$

The effective Hamiltonian for the hyperradial wave function of the pseudoharmonic potential is

$$
\hat{\mathcal{H}} = \frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu}\frac{(\ell+\frac{N-3}{2})(\ell+\frac{N-1}{2})}{r^2} + \frac{kr_0^2}{8}\left(\frac{r}{r_0^2} - \frac{r_0^2}{r}\right)^2.
\tag{53}
$$

In order to find $\langle r^{-2}\rangle$, we let $q = \ell$ in (52), then,

$$
\langle\Psi(\ell)|\frac{\partial\hat{\mathcal{H}}(\ell)}{\partial\ell}|\Psi(\ell)\rangle = \frac{\hbar^2}{2\mu}[2\ell + N - 2]\langle r^{-2}\rangle,
\tag{54}
$$

while,

$$
\frac{\partial E_{n_r,\ell}}{\partial\ell} = \frac{(2\ell+N-2)}{2\sqrt{(2\ell+N-2)^2 + \frac{\mu k r_0^4}{\hbar^2}}}\sqrt{\frac{\hbar^2 k}{\mu}}.
\tag{55}
$$

Therefore, the expectation values for $\langle r^{-2}\rangle$ (using HFT) in this case is:

$$
\langle r^{-2}\rangle = \frac{\sqrt{\frac{\mu k}{\hbar^2}}}{\sqrt{(2\ell+N-2)^2 + \frac{\mu k r_0^4}{\hbar^2}}}.
\tag{56}
$$

To find $\langle r^2\rangle$, we let $q=k$ in (52), then

$$\langle\Psi(k)|\frac{\partial\hat{\mathcal{H}}(k)}{\partial k}|\Psi(k)\rangle=\frac{1}{8}(\langle r^2\rangle+r_0^4\langle r^{-2}\rangle-2r_0^2)=\frac{\partial E_{n_r}(k)}{\partial k}, \tag{57}$$

where

$$\frac{\partial E_{n_r}(k)}{\partial k}=\frac{1}{2}\left[n_r+\frac{N}{4}-\frac{N-2}{4}+\frac{1}{4}\sqrt{(2\ell+N-2)^2+\frac{\mu k r_0^4}{\hbar^2}}\right]\sqrt{\frac{\hbar^2}{\mu k}}+\frac{1}{8}\frac{\sqrt{\frac{\mu k}{\hbar^2}}r_0^4}{\sqrt{(2\ell+N-2)^2+\frac{\mu k r_0^4}{\hbar^2}}}. \tag{58}$$

Therefore, the expectation values for $\langle r^2\rangle$ (using HFT) in this case is:

$$\langle r^2\rangle=4\left[n_r+\frac{N}{4}-\frac{N-2}{4}+\frac{1}{4}\sqrt{(2\ell+N-2)^2+\frac{\mu k r_0^4}{\hbar^2}}\right]\sqrt{\frac{\hbar^2}{\mu k}}=\frac{4}{k}E_{n_r}+r_0^2. \tag{59}$$

Finally, if we let $q=\mu$, then, (using HFT) we obtained the Virial theorem for pseudo-harmonic potential as follows:

$$\langle\Psi(\mu)|\frac{\partial\hat{\mathcal{H}}(\mu)}{\partial \mu}|\Psi(\mu)\rangle=-\frac{1}{\mu}\langle\hat{\mathcal{H}}-V\rangle=\frac{\partial E_{n_r}}{\partial \mu}, \tag{60}$$

where

$$\frac{\partial E_{n_r}}{\partial \mu}=-\frac{1}{2\mu}\left[n_r+\frac{N}{4}-\frac{N-2}{4}+\frac{1}{4}\sqrt{(2\ell+N-2)^2+\frac{\mu k r_0^4}{\hbar^2}}\right]\sqrt{\frac{\hbar^2 k}{\mu}}+\frac{k r_0^4}{8}\frac{\sqrt{\frac{\hbar^2 k}{\mu}}}{\sqrt{(2\ell+N-2)^2+\frac{\mu k r_0^4}{\hbar^2}}}. \tag{61}$$

With (61), (60) gives

$$\langle\hat{\mathcal{H}}-V\rangle=\frac{1}{2}\left(E_{n_r}+\frac{k r_0^2}{4}\right)-\frac{k r_0^4}{8}\Lambda, \tag{62}$$

where

$$\Lambda=\frac{\sqrt{\frac{\mu k}{\hbar^2}}}{\sqrt{(2\ell+N-2)^2+\frac{\mu k r_0^4}{\hbar^2}}}, \tag{63}$$

since

$$\hat{\mathcal{H}}=\langle T\rangle+\langle V\rangle=E_{n_r}, \tag{64}$$

then

$$\langle T\rangle=\langle V\rangle+\frac{k r_0^2}{4}(1-\Lambda r_0^2). \tag{65}$$

In a similar manner, the expectation values of the kinetic energy, the potential energy and the momentum-square are obtained respectively as:

$$\begin{aligned}\langle T\rangle &= \frac{1}{2}\left(E_{n_r}+\frac{kr_0^2}{4}\right)-\frac{kr_0^4}{8}\Lambda,\\ \langle p^2\rangle &= \mu E_{n_r}+\frac{\mu k r_0^2}{4}(1-r_0^2\Lambda),\\ \langle V\rangle &= \frac{1}{2}\left[E_{n_r}+\frac{kr_0^2}{4}(kr_0^2\Lambda-1)\right].\end{aligned}\tag{66}$$

It is pertinent to note that on extending the work of Popov [88], we can obtained the quantum mechanical average values of the energy, the kinetic energy and the momentum-square (as above) for the case of the pseudoharmonic potential in an arbitrary number of dimensions. Popov [88] obtained the following relations for expectation values of the energy and the kinetic energy from which we can also deduce the expectation value of the momentum-square as follows:

$$E_{n_r}=\langle V\rangle+\frac{1}{2}\left\langle r\frac{dV}{dr}\right\rangle,\qquad \langle T\rangle=\frac{1}{2}\left\langle r\frac{dV}{dr}\right\rangle,\qquad \langle p^2\rangle=\mu\left\langle r\frac{dV}{dr}\right\rangle.\tag{67}$$

From (66a), and noting from the pseudoharmonic potential that

$$\frac{dV}{dr}=\frac{k}{8}\left(2r-\frac{2r_0^4}{r^3}\right),$$

then energy is obtained as

$$\begin{aligned}E_{n_r}&=\langle V\rangle+\frac{1}{2}\left\langle r\frac{dV}{dr}\right\rangle=\frac{k}{8}(\langle r^2\rangle+r_0^4\langle r^{-2}\rangle-2r_0^2)+\frac{k}{8}(\langle r^2\rangle-r_0^4\langle r^{-2}\rangle)\\ &=\frac{k}{4}(\langle r^2\rangle-r_0^2)\\ &=\left[n_r+\frac{N}{4}-\frac{N-2}{4}+\frac{1}{4}\sqrt{(2\ell+N-2)^2+\frac{\mu k r_0^4}{\hbar^2}}\right]\sqrt{\frac{\hbar^2 k}{\mu}}-\frac{kr_0^2}{4}.\end{aligned}\tag{68}$$

From (66b), the expectation value of T is obtained as:

$$\begin{aligned}\langle T\rangle&=\frac{1}{2}\left\langle r\frac{dV}{dr}\right\rangle\\ &=\frac{k}{8}(\langle r^2\rangle-r_0^4\langle r^{-2}\rangle)=\frac{k}{8}\left(\frac{4E_{n_r}}{k}+r_0^2\right)-\frac{kr_0^4}{8}\frac{\sqrt{\frac{\mu k}{\hbar^2}}}{\sqrt{(2\ell+N-2)^2+\frac{\mu k r_0^4}{\hbar^2}}}\\ &=\frac{1}{2}\left(E_{n_r}+\frac{kr_0^2}{4}\right)-\frac{kr_0^4}{8}\Lambda.\end{aligned}\tag{69}$$

Finally from (66c), the expectation value of p^2 is obtained as:

$$\begin{aligned}\langle p^2\rangle &= \mu\left\langle r\frac{dV}{dr}\right\rangle \\ &= \frac{\mu k}{4}(\langle r^2\rangle - r_0^4\langle r^{-2}\rangle) = \frac{\mu k}{4}\left(\frac{4E_{n_r}}{k} + r_0^2\right) - \frac{\mu k r_0^4}{4}\frac{\sqrt{\frac{\mu k}{\hbar^2}}}{\sqrt{(2\ell+N-2)^2 + \frac{\mu k r_0^4}{\hbar^2}}} \\ &= \mu E_{n_r} + \frac{\mu k r_0^2}{4}(1 - r_0^2\Lambda). \end{aligned} \tag{70}$$

We have used the Hellmann–Feynman theorems to find some useful expectation values for pseudoharmonic potential in N-dimensions, also the virial theorem for this potential is obtained as $\langle T\rangle = \langle V\rangle + \frac{kr_0^2}{4}(1 - \Lambda r_0^2)$. The expectation values and the virial theorems (in (65)) depend on the dimensions as well as the parameters of the pseudoharmonic potential respectively.

5 Conclusion

In this paper, we have investigated some aspects of the pseudoharmonic potential in N dimensions. Analytical solutions are very important in the fields of Physics, these analytical solutions for the pseudoharmonic potential in N dimensions are obtained. The normalized hyperradial functions with the hyperspherical harmonics functions (complete energy eigenfunctions) and eigenvalues obtained for this potential depends on the values of the parameters of the potential and these results are dimension dependent.

The eigenfunctions and the eigenvalues reduce to known results if $N = 3$. We obtained the creation and annihilation operators for the wavefunction of the pseudoharmonic potential in N dimensions, and these operators satisfy the commutation relations of the generators of the dynamical group $SU(1,1)$ which are dimensions dependent. Also, the matrix elements of the different functions r^2, $r\frac{d}{dr}$, their difference and sum are obtained analytically from the ladder operators.

Using Hellmann–Feynman Theorem, some expectation values $\langle r^{-2}\rangle$, $\langle r^2\rangle$, $\langle\hat{\mathcal{H}}\rangle$, $\langle T\rangle$, $\langle V\rangle$ and $\langle p^2\rangle$ for the pseudoharmonic potential in an arbitrary number of dimensions are obtained. The results obtained in each case depends on the dimensions as well as the parameters of the potential. Finally, it is pertinent to note that (65) is the Virial theorems for the pseudoharmonic potential in an arbitrary number of dimensions, which is dimension dependent as well as the parameters of this potential. For example, if $r_0^2 = 0$ then, this potential becomes harmonic oscillator and then we obtained $\langle T\rangle = \langle V\rangle$ which is the usual Virial theorem for the harmonic oscillator potential in N dimensions. If $r_0^2 \neq 0$, then, the virial theorems as in (65) is preserved. The virial theorem for the pseudoharmonic potential depends on the dimension as well as the parameters of the potential.

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Appendix 1: Schrödinger Equation in an Arbitrary Number of Dimensions

Consider the motion of a particle in a spherically symmetric potential in N-dimensions,

$$\left[\frac{-\hbar^2}{2\mu}\Delta_N + V(r)\right]\Psi(r,\Omega_N) = E\Psi(r,\Omega_N), \tag{71}$$

where

$$\Delta_N \equiv \nabla_N^2 = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) - \frac{\Lambda_N^2(\Omega_N)}{r^2} \tag{72}$$

and $\Lambda_N^2(\Omega_N)$ is a partial differential operator on the unit sphere S^{N-1} called Laplace–Betrami operator, or grand orbital operator or hyperangular momentum operator [4, 16, 21, 64, 70, 79, 102]. This is analogously defined to a 3-dimensional angular momentum [9, 10];

$$\Lambda_N^2 = -\sum_{i\ge j}^{N}\Lambda_{ij}^2, \qquad \Lambda_{ij}^2 = x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}$$

for all Cartesian components x_i of the N-dimensional vector $(x_1, x_2, \ldots, x_N)$ or in another form of Barnea [14] and Rosati [93].

It is convenient to introduce the hyperspherical coordinates because of the spherical symmetry of the problem, which are defined as follows

$$\begin{aligned} x_1 &= r\cos\theta_1, \\ x_\alpha &= r\sin\theta_1\cdots\sin\theta_{\alpha-1}\cos\phi, \quad 2\le\alpha\le N-1, \\ x_N &= r\sin\theta_1\cdots\sin\theta_{N-2}\sin\phi. \end{aligned} \tag{73}$$

With $0\le\theta_k\le\pi$, $k=1,2,\ldots,N-2$, and $0\le\phi\le 2\pi$, where r is the polar variable and angular variables $\theta_1,\theta_2,\ldots,\theta_{N-2},\phi$ are the hyperangles. The Laplacian operator in polar coordinates $(r,\theta_1,\theta_2,\ldots,\theta_{N-2},\phi)$ of R^N is

$$\Delta_N = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) - \frac{\Lambda_N^2(\Omega_N)}{r^2}, \tag{74}$$

where,

$$\Lambda_N^2(\Omega_N) = \sum_{i=1}^{N-2}\left(\prod_{j=1}^{i}\sin\theta_j\right)^{-2}(\sin\theta_i)^{i+3-N}\frac{\partial}{\partial\theta_i}\left(\sin\theta_i^{N-i-1}\frac{\partial}{\partial\theta_i}\right) + \left(\prod_{j=1}^{N-2}\sin\theta_j\right)^{-2}\frac{\partial^2}{\partial\phi^2}. \tag{75}$$

Equation (75) is the non-radial part (hyperangular momentum operator or Laplace–Beltrami operator [9, 10, 15, 21, 70, 102]. This operator is known to fulfill

$$\Lambda_N^2(\Omega_N)Y_\ell^m(\Omega_N) = \ell(\ell+N-2)Y_\ell^m(\Omega_N), \tag{76}$$

where $\ell = 0,1,2,\ldots$ and $Y_\ell^m(\Omega_N)$ are the hyperspherical harmonics defined by [9, 10, 15, 53, 54, 70, 83, 85, 93]

$$Y_\ell^m(\Omega_N) = N_{\ell,m}\exp(im\phi)\prod_{k=1}^{N-2}C_{Z_k-Z_{k+1}}^{\alpha_k+Z_{k+1}}(\cos\theta_k)(\sin\theta_k)^{Z_{k+1}}, \tag{77}$$

with the normalization constant

$$N_{\ell,m}^2 = \frac{1}{2\pi} \prod_{k=1}^{N-2} \frac{(\alpha_k + Z_k)(Z_k + Z_{k+!})![\Gamma(\alpha_k + Z_{k+1})]^2}{\pi 2^{1-2\alpha_k - 2Z_{k+1}} \Gamma(2\alpha_k + Z_k + Z_{k+1})}. \tag{78}$$

Appendix 2: Ladder Operators

Consider the following expression:

$$\frac{d}{dr} R_{n_r,\ell}(r) = \frac{\mu_\ell}{r} R_{n_r,\ell}(r) - 2\lambda r R_{n_r,\ell}(r) + N_{n_r}^\lambda r^{\mu_\ell} \exp(-\lambda r^2) \frac{d}{dr} L_{n_r}^{\mu_\ell + \frac{N-2}{2}}(2\lambda r^2). \tag{79}$$

In order to evaluate the last term of (79), we consider the following derivative of the associated Laguerre functions [50]

$$x \frac{d}{dx} L_n^\alpha(x) = n L_n^\alpha(x) - (n+\alpha) L_{n-1}^\alpha(x), \tag{80}$$

with $x = 2\lambda r^2$ in (80), we have

$$\frac{r}{2} \frac{d}{dr} L_n^\alpha(2\lambda r^2) = n L_n^\alpha(2\lambda r^2) - (n+\alpha) L_{n-1}^\alpha(2\lambda r^2). \tag{81}$$

Putting this relation in (79) to obtain

$$\begin{aligned} &-\frac{d}{dr} R_{n_r,\ell}(r) + \frac{\mu_\ell}{r} R_{n_r,\ell}(r) - 2\lambda r R_{n_r,\ell}(r) + \frac{2n_r}{r} R_{n_r,\ell}(r) \\ &= \frac{2}{r}\left(n_r + \mu_\ell + \frac{N-2}{2}\right) \frac{N_{n_r}^\lambda}{N_{n_r-1}^\lambda} R_{n_r-1,\ell}(r), \end{aligned} \tag{82}$$

which with $N_{n_r}^\lambda$ becomes

$$\frac{1}{2}\left[-r\frac{d}{dr} - 2\lambda r^2 + 2n_r + \mu_\ell\right] R_{n_r,\ell}(r) = \sqrt{n_r\left(n_r + \mu_\ell + \frac{N-2}{2}\right)} R_{n_r-1,\ell}(r). \tag{83}$$

Similarly by considering the following expression:

$$\frac{d}{dr} R_{n_r,\ell}(r) - \frac{\mu_\ell}{r} R_{n_r,\ell}(r) + 2\lambda r R_{n_r,\ell}(r) = N_{n_r}^\lambda r^{\mu_\ell} \exp(-\lambda r^2) \frac{d}{dr} L_{n_r}^{\mu_\ell + \frac{N-2}{2}}(2\lambda r^2) \tag{84}$$

and to this end, we make use of this following associated Laguerre functions [50]:

$$x \frac{d}{dx} L_n^\alpha(x) = (n+1) L_{n+1}^\alpha(x) - (n+\alpha+1-x) L_{n-1}^\alpha(x), \tag{85}$$

which with $x = 2\lambda r^2$, becomes

$$\frac{r}{2} \frac{d}{dr} L_n^\alpha(2\lambda r^2) = (n+1) L_{n+1}^\alpha(2\lambda r^2) - (n+\alpha+1-2\lambda r^2) L_{n-1}^\alpha(2\lambda r^2) \tag{86}$$

and

$$\frac{1}{2}\left[r\frac{d}{dr} - 2\lambda r^2 + 2n_r + \mu_\ell + N\right]R_{n_r,\ell}(r) = \sqrt{(n_r+1)\left(n_r + \mu_\ell + \frac{N}{2}\right)}R_{n_r+1,\ell}(r). \quad (87)$$

The commutators can be evaluated as follow:

$$\begin{aligned}
&[\hat{\mathcal{L}}_-, \hat{\mathcal{L}}_+]R_{n_r,\ell}(r) \\
&\quad = \hat{\mathcal{L}}_-\{\hat{\mathcal{L}}_+ R_{n_r,\ell}(r)\} - \hat{\mathcal{L}}_+\{\hat{\mathcal{L}}_- R_{n_r,\ell}(r)\} \\
&\quad = \sqrt{(n_r+1)\left(n_r + \mu_\ell + \frac{N}{2}\right)}\{\hat{\mathcal{L}}_- R_{n_r+1,\ell}(r)\} \\
&\qquad - \sqrt{n_r\left(n_r + \mu_\ell + \frac{N-1}{2}\right)}\{\hat{\mathcal{L}}_+ R_{n_r-1,\ell}(r)\} \\
&\quad = \left(2n_r + \mu_\ell + \frac{N}{2}\right)R_{n_r,\ell}(r) = 2\ell_0 R_{n_r,\ell}(r),
\end{aligned} \quad (88)$$

where $\ell_0 = n_r + \frac{2\mu_\ell + N}{4}$ and $\hat{\mathcal{L}}_0 = \hat{n}_r + \frac{2\mu_\ell + N}{4}$.

Also,

$$\begin{aligned}
&[\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_-]R_{n_r,\ell}(r) \\
&\quad = \hat{\mathcal{L}}_0\{\hat{\mathcal{L}}_- R_{n_r,\ell}(r)\} - \hat{\mathcal{L}}_-\{\hat{\mathcal{L}}_0 R_{n_r,\ell}(r)\} \\
&\quad = \sqrt{n_r\left(n_r + \mu_\ell + \frac{N-1}{2}\right)}\{\hat{\mathcal{L}}_0 R_{n_r-1,\ell}(r)\} - \left(n_r + \frac{2\mu_\ell + N}{4}\right)\{\hat{\mathcal{L}}_- R_{n_r,\ell}(r)\} \\
&\quad = \sqrt{n_r\left(n_r + \mu_\ell + \frac{N-1}{2}\right)}\left\{\left(n_r - 1 + \frac{2\mu_\ell + N}{4}\right) - \left(n_r + \frac{2\mu_\ell + N}{4}\right)\right\}R_{n_r-1,\ell}(r) \\
&\quad = -\sqrt{n_r\left(n_r + \mu_\ell + \frac{N-1}{2}\right)}R_{n_r-1,\ell}(r) = -\ell_- R_{n_r-1,\ell}(r) = -\hat{\mathcal{L}}_- R_{n_r,\ell}(r),
\end{aligned} \quad (89)$$

and

$$\begin{aligned}
&[\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_+]R_{n_r,\ell}(r) \\
&\quad = \hat{\mathcal{L}}_0\{\hat{\mathcal{L}}_+ R_{n_r,\ell}(r)\} - \hat{\mathcal{L}}_+\{\hat{\mathcal{L}}_0 R_{n_r,\ell}(r)\} \\
&\quad = \sqrt{(n_r+1)\left(n_r + \mu_\ell + \frac{N}{2}\right)}\{\hat{\mathcal{L}}_0 R_{n_r+1,\ell}(r)\} - \left(n_r + \frac{2\mu_\ell + N}{4}\right)\{\hat{\mathcal{L}}_+ R_{n_r,\ell}(r)\} \\
&\quad = \sqrt{(n_r+1)\left(n_r + \mu_\ell + \frac{N}{2}\right)}\left\{\left(n_r + 1 + \frac{2\mu_\ell + N}{4}\right) - \left(n_r + \frac{2\mu_\ell + N}{4}\right)\right\}R_{n_r+1,\ell}(r) \\
&\quad = \sqrt{n_r\left(n_r + \mu_\ell + \frac{N}{2}\right)}R_{n_r+1,\ell}(r) = \ell_+ R_{n_r+1,\ell}(r) = \hat{\mathcal{L}}_+ R_{n_r,\ell}(r).
\end{aligned} \quad (90)$$

The following expressions can easily be obtained from the operators $\hat{\mathcal{L}}_{\mp}$ and $\hat{\mathcal{L}}_0$ as follows:

$$\begin{aligned}
&\{\hat{\mathcal{L}}_0(\hat{\mathcal{L}}_0-1)-\hat{\mathcal{L}}_+\hat{\mathcal{L}}_-\}R_{n_r,\ell}(r)\\
&\quad=\hat{\mathcal{L}}_0\{\hat{\mathcal{L}}_0-1\}R_{n_r,\ell}(r)-\hat{\mathcal{L}}_+\{\hat{\mathcal{L}}_-R_{n_r,\ell}(r)\}\\
&\quad=\left(n_r+\frac{2\mu_\ell+N}{4}\right)\left(n_r+\frac{2\mu_\ell+N}{4}-1\right)R_{n_r,\ell}(r)-n_r\left(n_r+\mu_\ell+\frac{N-2}{2}\right)R_{n_r,\ell}(r)\\
&\quad=\frac{2\mu_\ell+N}{4}\left\{\frac{2\mu_\ell+N-4}{4}\right\}R_{n_r,\ell}(r)=\gamma(\gamma-1)R_{n_r,\ell}(r)
\end{aligned}\tag{91}$$

and

$$\begin{aligned}
&\{\hat{\mathcal{L}}_0(\hat{\mathcal{L}}_0+1)-\hat{\mathcal{L}}_-\hat{\mathcal{L}}_+\}R_{n_r,\ell}(r)\\
&\quad=\hat{\mathcal{L}}_0\{\hat{\mathcal{L}}_0+1\}R_{n_r,\ell}(r)-\hat{\mathcal{L}}_-\{\hat{\mathcal{L}}_+R_{n_r,\ell}(r)\}\\
&\quad=\left(n_r+\frac{2\mu_\ell+N}{4}\right)\left(n_r+\frac{2\mu_\ell+N}{4}+1\right)R_{n_r,\ell}(r)\\
&\qquad-(n_r+1)\left(n_r+\mu_\ell+\frac{N}{2}\right)R_{n_r,\ell}(r)\\
&\quad=\gamma(\gamma-1)R_{n_r,\ell}(r).
\end{aligned}\tag{92}$$

The following matrix elements can be deduced as follow

$$\begin{aligned}
&\langle R_{m_r,\ell}(r)|r^2|R_{n_r,\ell}(r)\rangle\\
&\quad=\frac{1}{2\lambda}[\langle R_{m_r,\ell}(r)|2\hat{\mathcal{L}}_0|R_{n_r,\ell}(r)\rangle\\
&\qquad-\langle R_{m_r,\ell}(r)|\hat{\mathcal{L}}_+|R_{n_r,\ell}(r)\rangle-\langle R_{m_r,\ell}(r)|\hat{\mathcal{L}}_-|R_{n_r,\ell}(r)\rangle]\\
&\quad=\frac{1}{2\lambda}\left[\int_0^\infty\left(2n_r+\frac{2\mu_\ell+N}{2}\right)|R_{m_r,\ell}(r)R_{n_r,\ell}(r)|dr\right.\\
&\qquad\left.-\int_0^\infty\ell_+|R_{m_r,\ell}(r)R_{n_r+1,\ell}(r)|dr-\int_0^\infty\ell_-|R_{m_r,\ell}(r)R_{n_r-1,\ell}(r)|dr\right]\\
&\quad=\frac{1}{2\lambda}\left[\left(2n_r+\frac{2\mu_\ell+N}{2}\right)\delta_{m_r,n_r}-\ell_+\delta_{m_r,n_r+1}-\ell_-\delta_{m_r,n_r-1}\right]
\end{aligned}\tag{93}$$

and

$$\begin{aligned}
&\langle R_{m_r,\ell}(r)|r\frac{d}{dr}|R_{n_r,\ell}(r)\rangle\\
&\quad=\langle R_{m_r,\ell}(r)|\hat{\mathcal{L}}_+|R_{n_r,\ell}(r)\rangle\\
&\qquad-\langle R_{m_r,\ell}(r)|\hat{\mathcal{L}}_-|R_{n_r,\ell}(r)\rangle-\langle R_{m_r,\ell}(r)|\frac{N}{2}|R_{n_r,\ell}(r)\rangle\\
&\quad=\int_0^\infty\ell_+|R_{m_r,\ell}(r)R_{n_r+1,\ell}(r)|dr-\int_0^\infty\ell_-|R_{m_r,\ell}(r)R_{n_r-1,\ell}(r)|dr\\
&\qquad-\int_0^\infty\frac{N}{2}|R_{m_r,\ell}(r)R_{n_r,\ell}(r)|dr=\ell_+\delta_{m_r,n_r+1}-\ell_-\delta_{m_r,n_r-1}-\frac{N}{2}\delta_{m_r,n_r}.
\end{aligned}\tag{94}$$

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